ES. $\int$ curvilinei su curve orientate

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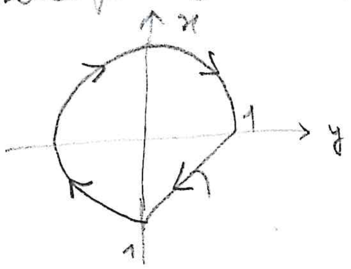
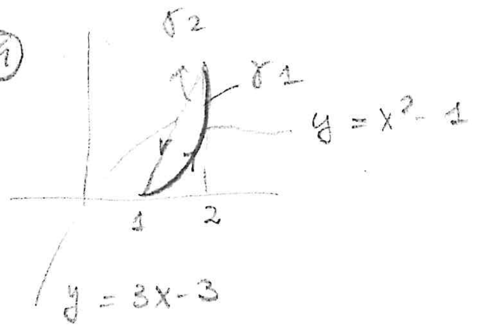
① Calcolare l'$\int$ di $\omega = -y\,dx + x\,dy$ lungo γ in figura



② Sia γ la curva orientata di pto iniziale $(0,0,-\sqrt{2})$ e individuata da $x^2+y^2+z^2=2$ $y=x\geq 0$
Calcolare il lavoro di $F(x,y,z)=(z,z,-x-y)$

③ Sia $\gamma(t)=(\cos t, \operatorname{sen} t, 2)$ $t\in[0,2\pi]$.
Calcolare $\displaystyle\int_{(\gamma,T)} (y\,dx + z\,dy + x\,dz)$

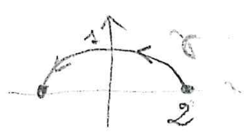
④



$$\int_{(\gamma,T)} (x+y)\,dx - (x-y)\,dy$$

(anche usando G.G.)

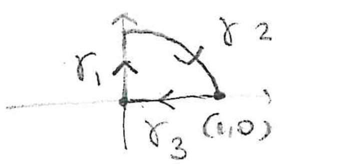
⑤



(ellisse) Calcolare il lavoro di $F=(y^2, x^2)$

⑥ Calcolare l'area racchiusa ~~dalla~~ curva asteroide $\gamma(t)=(\cos^3 t, \operatorname{sen}^3 t)$ $t\in[0,2\pi]$ usando G.G.

⑦



$\gamma=\gamma_1+\gamma_2+\gamma_3$

$$\int_\gamma \frac{3x}{x^2+y^2+1}\,dx - \frac{3y}{x^2+y^2+1}\,dy$$

usando la def e usando G.G.

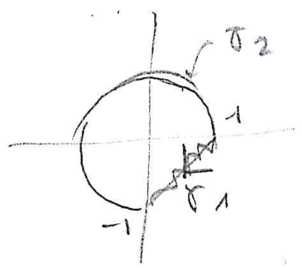
⑧ Sia $D=\{(x,y) \mid x^2+y^2\leq 1,\ x\geq |y|\}$
Calcolare il lavoro di $F(x,y)=(y,-x)$ lungo $+\partial D$ usando la def e usando G.G.

es. svolti | curvilinei
su curve orientate

Cu

M. Mannucci (?)

(1)



$\gamma = \gamma_1 + \gamma_2$

$\gamma_1(t) = (1-t, -t) \quad t \in [0,1]$

$y = x - 1$ compatibile con l'orientamento

$x = 1 + y$

$\gamma_2(t) = (\cos t, \operatorname{sen} t) \quad t \in [0, \frac{3}{2}\pi]$ non comp. con l'or.

$$\int_{(\gamma,T)} -y\,dx + x\,dy = \int_{(\gamma_1 T)} \text{---} + \int_{(\gamma_2,T)} \text{---} =$$

$$= \int_0^1 [(+t)(-1) + (1-t)(-1)]\,dt - \int_0^{3/2\pi} (-\operatorname{sen} t)(-\operatorname{sen} t) + \cos t \cos t \, dt$$

$$= 1 + \frac{3}{2}\pi$$

(usando il t. di Gauss-Green

$$\int_\gamma -y\,dx + x\,dy = \int_D \frac{\partial}{\partial x}(x)\,dx\,dy - \int_D \frac{\partial}{\partial y}(-y)\,dx\,dy =$$

$$= \mu(D) + \mu(D) = 2\left(\frac{3}{4}\pi + \frac{1}{2}\right) = \frac{3}{2}\pi + 1 \quad)$$

(2) parametrizziamo usando le coord. sferiche

$$\begin{cases} x = \sqrt{2}\,\operatorname{sen}\varphi \cos\theta \\ y = \sqrt{2}\,\operatorname{sen}\varphi \operatorname{sen}\theta \\ z = \sqrt{2}\cos\varphi \end{cases} \qquad \begin{matrix} \theta \in]0, 2\pi[\\ \varphi \in]0, \pi[\end{matrix}$$

$0 < x = y \iff \cos\theta = \operatorname{sen}\theta > 0 \iff \theta = \frac{\pi}{4}$

$$\gamma \begin{cases} x = \sqrt{2}\,\operatorname{sen}\varphi \\ y = \sqrt{2}\,\operatorname{sen}\varphi \\ z = \sqrt{2}\cos\varphi \end{cases}$$

pto iniziale $(0,0,-\sqrt{2})$

$\gamma(0) \neq (0,0,-\sqrt{2}) \qquad \gamma(\pi) = (0,0,-\sqrt{2})$

$\operatorname{sen}\varphi > 0 \iff \varphi \in]0, \pi[\qquad \gamma$ non è comp.

Cu 2

lavoro $= -\int_0^{\pi} \sqrt{2}\cos\varphi\cdot(\sqrt{2}\cos\varphi)+\sqrt{2}\cos\varphi(\sqrt{2}\cos\varphi)$

$+(-\sqrt{2}\,2\,\mathrm{sen}\,\varphi)(-\mathrm{sen}\,\varphi)\,d\varphi =$

$$= -\int_0^{\pi} (2\sqrt{2}\cos^2\varphi + 2\sqrt{2}\,\mathrm{sen}^2\varphi) = -2\sqrt{2}\,\pi$$

(3) $\int_0^{2\pi}(\mathrm{sen}\,t\,(-\mathrm{sen}\,t)+2\cos t)\,dt = -\frac{1}{2}(t-\mathrm{sen}\,t\cos t)+\mathrm{sen}\,t\Big]_{t=0}^{2\pi} = \dots$

(4) $\gamma_1(t)=(t,t^2-1)$, $t\in[1,2]$ $\qquad \gamma_2(t)=(t,3t-3)$, $t\in[1,2]$ (non conc. con l'orien.)

$$\int_{(\gamma_1,T)}\dots = \int_1^2 [(t+t^2-1)-(t-t^2+1)2t]\,dt = \dots$$

$$\int_{(\gamma_2,T)} = -\int_1^2[(t+3t-3)-(t-3t+3)\cdot 3]\,dt = \dots$$

II modo usando G-G. $\left(\iint_D \frac{\partial f}{\partial x} = \int_{\partial^+D} f\,dy \qquad \iint_D \frac{\partial \varrho}{\partial y} = -\int_{\partial D^+}\varrho\,dx\right)$

$$\int_{(\gamma,T)=\partial^+D}(x+y)\,dx - \int_{(\gamma,T)=\partial^+D}(x-y)\,dy = -\iint_D\frac{\partial}{\partial y}(x+y)\,dx\,dy - \int_D \frac{\partial}{\partial x}(x-y)\,dx\,dy$$

$$= -\iint_D(1+1)\,dx\,dy$$

$$= -2\int_1^2\left(\int_{x^2-1}^{3x-3}dy\right)dx$$

(5) $\gamma(t)=(2\cos t,\ \mathrm{sen}\,t)$, $t\in[0,\pi]$

$\frac{x^2}{4}+y^2=1 \qquad F=(y^2,x^2)$

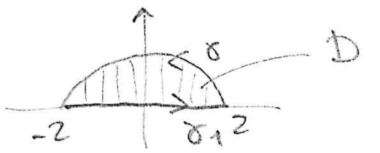
$$\int_\gamma y^2dx+x^2dy = \int_0^\pi[\mathrm{sen}^2t\,(-2\,\mathrm{sen}\,t)+4\cos^2t\,\cos t]\,dt =$$

$$= -2\int_0^\pi \mathrm{sen}^3t + 4\int_0^\pi\cos^3t\,dt = \dots$$

$\mathrm{sen}^3t = \mathrm{sen}\,t\,(1-\cos^2t) \qquad \cos^3t = \cos t\,(1-\mathrm{sen}^2t)$

II modo usando GG

"Chiudiamo il percorso" nel seg modo



$$\int_{\gamma\cup\gamma_2} y^2\,dx + x^2\,dy = -\iint_D \frac{\partial}{\partial y}(y^2)\,dx\,dy + \int_D \frac{\partial}{\partial x}(x^2)\,dx\,dy =$$

$$= \iint_D (2x-2y)\,dx\,dy = \dots$$

↑ coord. polari modificate

$\Downarrow$

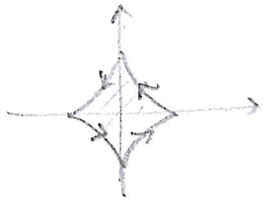
$$\int_\gamma y^2dx + x^2dy = \iint_D \text{---} \quad - \underbrace{\left(\int_{(\gamma_1,T)} y^2dx + x^2\,dy\right)}_{=0}$$

Infatti

$\gamma_1 = (t,0) \quad t\in[-2,2]$

$$\int_{(\gamma_1,T)} y^2\,dx + x^2\,dy = \int_{-2}^{2}(0+0)\,dt = 0$$

⑥



$$\text{area } D = \int_{\partial D^+} x\,dy = \int_0^{2\pi} \cos^3 t \; 3\,\text{sen}^2 t \cos t\, dt = \dots$$

$$= 3\int_0^{2\pi} \cos^4 t \; \text{sen}^2 t\, dt =$$

se usiamo $\quad \text{area } D = \frac{1}{2}\int_{\partial D^+}(x\,dy - y\,dx) =$

$$= \frac{1}{2}\int_0^{2\pi}(\cos^4 t\; 3\,\text{sen}^2 t + 3\cos^2 t\;\text{sen}^4 t)\,dt =$$

$$= \frac{3}{2}\int_0^{2\pi} \cos^2 t\;\text{sen}^2 t\, dt = \frac{3}{2}\int_0^{2\pi}\left(\frac{\text{sen}(2t)}{2}\right)^2 dt = \dots$$

(7) $\gamma_1(t)=(t,0)\quad t\in[0,1]$

$\gamma_2(t)=(\cos t, \operatorname{sen} t)\quad t\in[0,\pi/2]$

$\gamma_3(t)=(0,t)\quad t\in[0,1]$

nessuna compatibile

Cu

I modo

$$\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} = -\int_0^1 \frac{3t}{t^2+1}\,dt +$$

$$-\int_0^{\pi/2}\left[\frac{3\cos t}{2}(-\operatorname{sen} t) - \frac{3\operatorname{sen} t}{2}\cos t\right]dt - \int_0^1 \frac{-3t}{t^2+1}\,dt = \dots$$

II modo GG

$$\iint_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)dx\,dy = \int_{+\partial D} f_1\,dx + f_2\,dy$$

$$-\int_\gamma \text{---}\,dx + \text{---}\,dy = \iint_D \left(\frac{\partial}{\partial x}\left(\frac{-3y}{x^2+y^2+1}\right) - \frac{\partial}{\partial y}\left(\frac{3x}{x^2+y^2+1}\right)\right)dx\,dy$$

$$= \iint_D \left(\frac{6xy}{(x^2+y^2+1)^2} + \frac{6xy}{(x^2+y^2+1)^2}\right)dx\,dy =$$

$$= \iint_D \frac{12xy}{(x^2+y^2+1)^2}\,dx\,dy \underset{\substack{\uparrow\\ \text{coord. polari}}}{=} \int_0^{\pi/2}\int_0^1 \frac{12\rho^3\cos\theta\operatorname{sen}\theta}{(\rho^2+1)^2}\,d\rho\,d\theta =$$

$$= \left(\int_0^{\pi/2} 12\cos\theta\operatorname{sen}\theta\,d\theta\right)\int_0^1 \frac{\rho^3}{\rho^4+2\rho^2+1}\,d\rho$$

dove $\frac{\rho^3}{(\rho^2+1)^2} =$

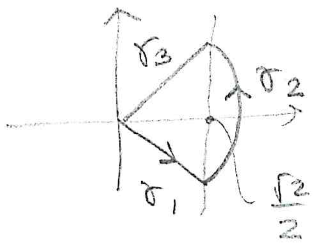
$$= \frac{1}{4}\,\frac{4\rho^3+4\rho}{\rho^4+2\rho^2+1} - \frac{\rho}{(\rho^2+1)^2}$$

↓ ha primitiva il $\log(\text{---})$

↳ $\int$ immediato

Cu 5

⑧



$\gamma_1(t) = (t, -t) \quad t \in [0, \sqrt{2}/2]$

$\gamma_2(t) = (\cos t, \operatorname{sen} t) \quad t \in [-\frac{\pi}{4}, \frac{\pi}{4}]$

$\gamma_3(t) = (t, t) \quad t \in [0, \pi/2]$ (non comp. con l'orient.)

$$\int_\gamma y\,dx - x\,dy = \underbrace{\cancel{\int_0^{\sqrt{2}/2} (-t + t)\, dt}}_{=0}\ \gamma_1 + \underbrace{\int_{-\pi/4}^{\pi/4} (-\operatorname{sen}^2 t - \cos^2 t)\, dt}_{= -\pi/2}$$

$$- \int_0^{\sqrt{2}/2} \underbrace{\cancel{(t - t)}}_{=0} dt = -\frac{\pi}{2}$$

II modo usando G.G.

$$\int_{+\partial D} f_1\, dx + f_2\, dy = \iint_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy =$$

$$= \iint_D -1 -1 \; dx\,dy = -2\,\mu(D) = -2\,\frac{\pi}{4} = -\frac{\pi}{2}.$$