ES. campi vett conservativi o forme differenziali esatte

F1

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Stabilire se le seg. forme diff. sono esatte (e se lo sono trovare una primitiva)

1) $\omega = 2y\,dx - \operatorname{sen} x\,dy$

2) $\omega = y e^x (x+1)\,dx + x e^x\,dy$

3) $\omega = \frac{1}{y}\,dx - \frac{x}{y^2}\,dy$

Stabilire se i seg. campi vett. sono conserv. (e se lo sono determ. un potenziale)

4) $F(x,y) = \left(y x^{y-1};\ y-1+x^y \log x\right)$

5) $F(x,y) = \left(y+\log(x+4),\ x+e^{-y}\right)$
calcolare il lavoro lungo $\gamma = \{(x,y) \mid y = x^3,\ 0 \leq x \leq 3\}$
di primo estremo $(0,0)$

6) $F(x,y) = \left(\frac{y}{\sqrt{xy}},\ \frac{x}{\sqrt{xy}}\right)$

7) $F(x,y) = \left(\frac{xy^2}{1+x^2y^2},\ \frac{x^2y}{1+x^2y^2}\right)$ Calcolare il lavoro lungo la curva $\gamma(t) = (\cos t - 1,\ 2\operatorname{sen} t + 4)$ $t \in [0, 2\pi)$.

8) $F(x,y,z) = \left(-\frac{z}{(x+y^2)^2},\ -\frac{2xy}{(x+y^2)^2} + \frac{1}{z},\ \frac{1}{x+y^2} - \frac{y}{z^2} + 2z\right)$

Calcolare un potenziale di F in $A = \{(x,y,z) \mid x,y,z > 0\}$

9) $F(x,y,z) = \left(\frac{2xy}{x^4+y^2},\ \frac{-x^2}{x^4+y^2}\right)$ $A = \mathbb{R}^2 \setminus \{(0,0)\}$

10) Data la forma diff.
$\omega(x,y,z) = xz\,dx + f_2(x,y,z)\,dy + \frac{c}{2}x^2\,dz$ $c \in \mathbb{R}$, $f_2 \in C^1$
stabilire se per qualche α, f_2 ω è esatta
e trovarne un potenziale

F_2

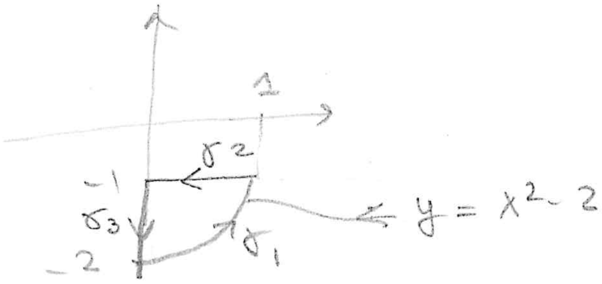
11) Data la forma diff

$$\omega = \frac{x}{x^2+y^2}\,dx + \left(\frac{y}{x^2+y^2} + x^2\right)dy. \quad \text{Calcolare} \int_{(\gamma,\Gamma)} \omega$$

γ è la parabola $y = 4-x^2$ $x \in [-2,1]$ I estremo $(-2,0)$

12) $$\omega = \left(\frac{1}{\sqrt{x-2y}} + \frac{1}{x^2+1}\right)dx + \left(\frac{-2}{\sqrt{x-2y}} + x\right)dy$$

Calcolare $\int_{(\gamma,\Gamma)} \omega$



13) $\omega =$ sen $z\,dx + z\,dy + g(x,y,z)\,dz$ $\quad g \in C^\infty$

$\exists$? g tale che ω è esatta

Trovare una primitiva $\hat{f}$ tale che $f(0,0,0) = 1$

1) $\omega : \mathbb{R}^2 \to (\mathbb{R}^2)^*$ $\quad \omega \in C^1$

poiché $\frac{\partial}{\partial y}(2y) = 2 \neq \frac{\partial}{\partial x}(-\text{sen} x)$ $\Rightarrow$ ω non è chiusa

$\Downarrow$

ω non è esatta

2) $\omega : \mathbb{R}^2 \to (\mathbb{R}^2)^*$ $\quad \omega \in C^1$

poiché $\frac{\partial}{\partial y}\left(y e^x (x+1)\right) = e^x(x+1) = \frac{\partial}{\partial x}(e^x x)$

ω è chiusa, $\mathbb{R}^2$ è stellato rispetto a tutti i suoi pti $\Rightarrow$ ω è esatta

Cerco una funzione $f : \mathbb{R}^2 \to \mathbb{R}$: $df = \omega$

cioè

$$\begin{cases} \frac{\partial f}{\partial x} = y e^x (x+1) \quad (*) \\ \frac{\partial f}{\partial y} = e^x x \end{cases} \Rightarrow f = y e^x x + c(x)$$

ove $\frac{\partial f}{\partial x} = y e^x (x+1) + c'(x)$

confrontando con $(*)$ $\Rightarrow$ $c'(x) = 0$

$\Downarrow$

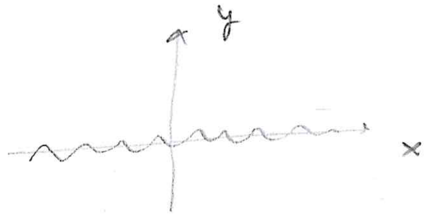
$c(x) = $ costante

scelgo x es. $c(x) \equiv 0$.

$\Downarrow$

$f(x,y) = y e^x (x+1)$ è una primitiva di ω.

3) $\omega : \mathbb{R}^2 \setminus \{y = 0\} \to (\mathbb{R}^2)^*$



$\frac{\partial}{\partial y}\left(\frac{1}{y}\right) = -\frac{1}{y^2} = \frac{\partial}{\partial x}\left(-\frac{x}{y^2}\right)$

ω è chiusa ma l'insieme non è stellato rispetto a nessun pto e non è semplic. connesso, non possiamo stabilire a priori se si tratta di una forma diff. esatta

D'altra parte, se esistesse f : $df = \omega$

allora $\forall\, y \neq 0$

$$\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{1}{y} \\ \dfrac{\partial f}{\partial y} = -\dfrac{x}{y^2} \quad (*) \end{cases} \longrightarrow f = \frac{x}{y} + c(y)$$

$\downarrow$ derivando risp. ad y

$$\frac{\partial f}{\partial y} = -\frac{x^2}{y} + c'(y)$$

confrontando con (*) si ha $c'(y) =$ costante.

Allora

$f(x,y) = -\dfrac{x^2}{y}$ è una funz. di cv in $\mathbb{R}^2 \setminus \{y=0\}$.

4) $F:]0,+\infty[\times \mathbb{R} \to \mathbb{R}^2$ dominio è stellato

$$F(x,y) = (y\, e^{(y-1)\log x},\ y-1+e^{y\log x}\log x)$$

$$\frac{\partial}{\partial y}\left(y\, e^{(y-1)\log x}\right) = e^{(y-1)\log x} + y\, e^{(y-1)\log x} \log x \ ) =$$

$$\frac{\partial}{\partial x}\left(y-1+e^{y\log x}\log x\right) = e^{y\log x}\,\frac{y}{x}\log x + e^{y\log x}\,\frac{1}{x}$$

dove $e^{(y-1)\log x} = e^{y\log x}\, e^{-\log x} = e^{y\log x}\,\dfrac{1}{x}$

$\Downarrow$

F è conservativo. Cerco U (potenziale di F):

$$\begin{cases} \dfrac{\partial U}{\partial x} = y\, e^{(y-1)\log x} \quad \left(= y\, e^{y\log x} \cdot \dfrac{1}{x}\right) \quad (*) \\ \dfrac{\partial U}{\partial y} = y-1+e^{y\log x}\log x \end{cases} \Rightarrow U(x,y) = \frac{y^2}{2} - y + e^{y\log x} + c(x)$$

derivando rispetto ad x

$$\frac{\partial U}{\partial x} = e^{y\log x} \cdot \frac{y}{x} + c'(x),$$ confrontando con (*)

si ha $c'(x) = 0 \Rightarrow c(x) =$ costante.

$\Downarrow$

$$U(x,y) = \frac{y^2}{2} - y + e^{y\log x}$$

5) $F : \underbrace{]-4,+\infty[}_{\text{stellato}} \times \mathbb{R} \longrightarrow \mathbb{R}^2$

$\frac{\partial F_1}{\partial y} = 1 = \frac{\partial F_2}{\partial x} \Rightarrow$ F chiuso $\Rightarrow$ F è cons.

Cerco U potenz. : $\begin{cases} \frac{\partial U}{\partial x} = y + \log(x+4) \\ \frac{\partial U}{\partial y} = x + e^{-y} \end{cases}$

(Calcolo a parte una primitiva (rispetto ad x) di $\log(x+4)$

$$\int_0^x \log(t+4)\,dt \underset{\substack{\uparrow \\ \text{x parti}}}{=} (t+4)\log(t+4)\Big]_0^x - \int_0^x t+4 \frac{1}{t+4}\,dt =$$

$= (x+4)\log(x+4) - 4\log 4 - x$)

Allora

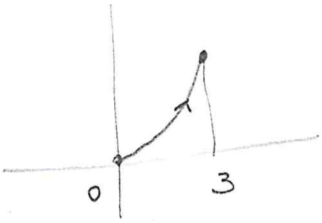
$$U(x,y) = xy + (x+4)\log(x+4) - x + c(y)$$

derivando rispetto ad y

$x + e^{-y} = \frac{\partial U}{\partial y} = x + c'(y) \Leftrightarrow c'(y) = e^{-y} \quad c(y) = -e^{-y}$

$\Downarrow$

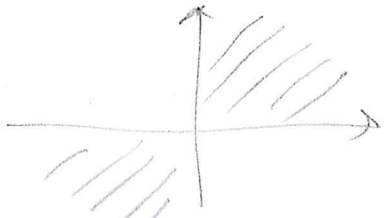
$$U(x,y) = x(y-1) + (x+4)\log(x+4) - e^{-y}$$



I estremo $= (0,0)$
II estremo $= (3,27)$

lavoro $= \int_{(\gamma,T)} F = U(3,27) - U(0,0) = \dots$

6) $F : \underbrace{\{(x,y) \mid xy>0\}}_{B} \longrightarrow \mathbb{R}^2$



no stellato
no sempl. connesso

Considero $F\big|_{\{(x,y) \mid x>0,\, y>0\}} = F\big|_A$

$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ quindi $F\big|_A$ è conserv.

cerco un potenz. in A

$$\begin{cases} \frac{\partial U}{\partial x} = \frac{y}{\sqrt{xy}} = \frac{1}{\sqrt{x}}\sqrt{y} \\ \frac{\partial U}{\partial y} = \frac{1}{\sqrt{y}}\sqrt{x} \end{cases} \Rightarrow U(x,y) = 2\sqrt{x}\sqrt{y} + c(y)$$

$\downarrow$ derivo

$$\frac{\partial U}{\partial y} = \frac{\sqrt{x}}{\sqrt{y}} + c'(y) = \frac{\sqrt{x}}{\sqrt{y}}$$

$\Downarrow$

$c'(y)=0$ scelgo $c(y) \equiv 0$

Allora

$U(x,y) = 2\sqrt{xy}$ è un potenz. in A

si verifica che è anche un potenz. in B

7) $F : \mathbb{R}^2 \to \mathbb{R}^2$. Si verifica che F è chiuso $\Rightarrow$ F è cons.

$$\begin{cases} \frac{\partial U}{\partial x} = \frac{xy^2}{1+x^2y^2} \\ \frac{\partial U}{\partial y} = \frac{x^2y}{1+x^2y^2} \end{cases} \Rightarrow U(x,y) = \frac{1}{2}\log(1+x^2y^2) + c(y)$$

$\downarrow$

$$\frac{\partial U}{\partial y} = \frac{x^2y}{1+x^2y^2} + c'(y) \Rightarrow c(y) \equiv 0 \text{ x es.}$$

$U(x,y) = \frac{1}{2}\log(1+x^2y^2)$

ed essendo γ chiusa (γ è un ellisse) $\Rightarrow \int_{(\gamma,T)} F = 0$

8) $U : A \subset \mathbb{R}^3 \to \mathbb{R}^3$:

$$\frac{\partial U}{\partial x} = -\frac{z}{(x+y^2)^2} \Rightarrow U(x,y,z) = \frac{z}{x+y^2} + c_1(y,z)$$

deriviamo rispetto ad y:

$$\frac{\partial U}{\partial y} = -\frac{2yz}{(x+y^2)^2} + \frac{\partial c_1}{\partial y}(y,z) = -\frac{2yz}{(x+y^2)^2} + \frac{1}{z} \Rightarrow \frac{\partial c_1}{\partial y}(y,z) = \frac{1}{z}$$

$\Downarrow$

$$c_1(y,z) = \frac{y}{z} + c_2(z)$$

$\Downarrow$

$$U(x,y,z) = \frac{z}{x+y^2} + \frac{y}{z} + c_2(z)$$

deriviamo rispetto a z:

$$\frac{\partial U}{\partial z} = \frac{1}{x+y^2} - \frac{y}{z^2} + C_2'(z) = -\frac{y}{z^2} + 2z + \frac{1}{x+y^2}$$

$$\Rightarrow C_2'(z) = 2z \Rightarrow C_2(z) = z^2 \quad (\text{x es.})$$

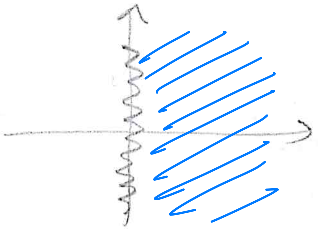
$$\Downarrow$$

$$U(x,y,z) = \frac{z}{x+y^2} + \frac{y}{z} + z^2$$

9)
$$\left.\begin{aligned} \frac{\partial F_1}{\partial y} &= \frac{2x(x^4+y^2) - 2xy\,2y}{(x^4+y^2)^2} \\ \frac{\partial F_2}{\partial x} &= \frac{-2x(x^4+y^2) + x^2 4x^3}{(x^4+y^2)^2} \end{aligned}\right) = \Rightarrow F \text{ è chiusa}$$

$x>0$

In particolare F è esatta in $\mathbb{R}^2 \setminus \{\cancel{x=0}\}$ (che è semp. connesso)

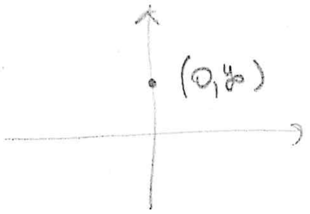


$$\frac{\partial U}{\partial x} = \frac{2xy}{x^4+y^2} = \frac{2xy}{x^4\left(1+\left(\frac{y}{x^2}\right)^2\right)} = \frac{2y}{x^3}\cdot\frac{1}{1+\left(\frac{y}{x^2}\right)^2}$$

$$\Rightarrow U(x,y) = -\text{arctg}\,\frac{y}{x^2} + C(y)$$

$$\frac{\partial U}{\partial y}(x,y) = -\frac{1}{1+\left(\frac{y}{x^2}\right)^2}\cdot\frac{1}{x^2} \overset{+C'(y)}{=} \frac{-x^2}{x^4+y^2} = F_1 \Rightarrow C'(y) = 0$$

$U(x,y) = -\text{arctg}\,\frac{y}{x^2}$ potenziale in $\mathbb{R}^2 \setminus \{x=0\}$. (verifica)



Lo estendiamo
$$\tilde U(x,y) = \begin{cases} U(x,y) & x \neq 0 \\ -\frac{\pi}{2} & x = 0 \end{cases}$$

Proviamo che $\tilde U$ ha le derivate parziali in $(0,y_0)$ con $y_0 \neq 0$

$$\frac{\partial \tilde U}{\partial x}(0,y_0) = \lim_{h\to 0} \frac{\tilde U(h,y_0) - \tilde U(0,y_0)}{h} =$$

$$= \lim_{h\to 0} \frac{-\text{arctg}\,\frac{y_0}{h^2} + \frac{\pi}{2}}{h} \underset{\substack{\uparrow \\ \text{arctg}\,x + \text{arctg}\,\frac{1}{x} = \frac{\pi}{2}}}{=} \lim_{h\to 0} \frac{\text{arctg}\left(\frac{h^2}{y_0}\right)}{h} = \text{arctg}(0) = 0$$

$$\frac{\partial \tilde{U}}{\partial y}(0,y_0) = \lim_{h\to 0} \frac{U(0,y_0+h) - U(0,y_0)}{h} = 0$$

allora
$\tilde{U}(x,y)$ e' differenziabile in $(0,y_0)$

(infatti

$$\frac{\tilde{U}(h, y_0+k) - \overset{= -\frac{\pi}{2}}{\tilde{U}(0,y_0)} - \overset{=0}{\frac{\partial \tilde{U}}{\partial x}(0,y_0)}\, h - \overset{=0}{\frac{\partial \tilde{U}}{\partial y}(0,y_0)}\, k}{\sqrt{h^2+k^2}} =$$

$$= \frac{-\operatorname{arctg}\left(\frac{y_0+k}{h^2}\right) + \frac{\pi}{2}}{\sqrt{h^2+k^2}} = \frac{\operatorname{arctg}\left(\frac{h^2}{y_0+k}\right)}{\sqrt{h^2+k^2}} =$$

$$= \frac{\operatorname{arctg}\left(\frac{h^2}{y_0+k}\right) - \operatorname{arctg}\left(\frac{0}{y_0+k}\right)}{\sqrt{h^2+k^2}} \xrightarrow[(h,k)\to(0,0)]{} 0 \quad)$$

10) $\omega : \mathbb{R}^3 \to (\mathbb{R}^3)^*$ $\quad \mathbb{R}^3$ e' stellato

quindi ω e' esatta $\Leftrightarrow$ ω e' chiusa

$$\left.\begin{aligned} &\frac{\partial f_1}{\partial y} = 0 = \frac{\partial f_2}{\partial x} \quad e \quad \frac{\partial f_1}{\partial z} = x = \frac{\partial f_3}{\partial x} = Cx \\ &e \quad \frac{\partial f_3}{\partial y} = 0 = \frac{\partial f_2}{\partial z} \end{aligned}\right\} \Leftrightarrow \begin{cases} C = 1 \\ \frac{\partial f_2}{\partial x} = \frac{\partial f_2}{\partial z} = 0 \end{cases}$$

quindi
$f_2 = f_2(y)$

Una primitiva di ω:

$$\begin{cases} \frac{\partial f}{\partial x} = xz \\ \frac{\partial f}{\partial y} = f_2(y) \\ \frac{\partial f}{\partial z} = \frac{x^2}{2} \end{cases} \qquad f(x,y,z) = \frac{x^2}{2}z + c(y,z)$$

$$\downarrow$$

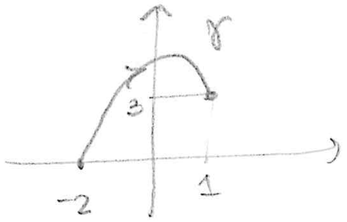
$$\frac{\partial f}{\partial y} = \frac{\partial c}{\partial y}(y,z) = f_2(y)$$

$$\frac{\partial f}{\partial z} = \frac{x^2}{2} + \frac{\partial c}{\partial z}(y,z) = \frac{x^2}{2}$$

quindi $\frac{\partial c}{\partial z} = 0 \quad \frac{\partial c}{\partial y} = f_2(y) \Rightarrow c(y) = \int_0^y f_2(s)$

$$\Rightarrow f(x,y,z) = \frac{x^2 z}{2} + \int_0^y f_2(s)\, ds$$

11) osserv che ω non e' chiusa $\Rightarrow$ ω non e' esatta



$\gamma(t) = (t, 4-t^2)$
$t \in [-2,1]$

$$\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2} + \underset{\uparrow}{x^2}\right) = \frac{-2xy}{(\dots)^2} + 2x$$

problema viene da questo

Allora uso la linearità di ω e scrivo

$$\omega = \underbrace{\frac{x}{x^2+y^2}\,dx + \frac{y}{x^2+y^2}\,dy}_{\omega_1 \text{ chiusa}} + \underbrace{x^2\,dy}_{\omega_2}$$

in un insieme stellato che contiene γ, quindi e' esatta

una primitiva e'
$f(x,y) = \frac{1}{2}\log(x^2+y^2)$

ora $\displaystyle\int_{(\gamma,T)} \omega_1 = f(1,3) - f(-2,0) = \frac{1}{2}\log 10 - \frac{1}{2}\log 4 = \frac{1}{2}\log\frac{5}{2}$

e $\displaystyle\int_{(\gamma,T)} \omega_2 = \int_{-2}^{1} t^2(-2t)\,dt = -\frac{t^4}{2}\Big]_{-2}^{1}$

Allora

$$\int_{(\gamma,T)} \omega = \int_{(\gamma,T)} \omega_1 + \int_{(\gamma,T)} \omega_2 .$$

12) $\frac{\partial}{\partial x}\left(\frac{-2}{\sqrt{x-2y}} + x \right) = \frac{1}{(x-2y)^{3/2}} + 1$

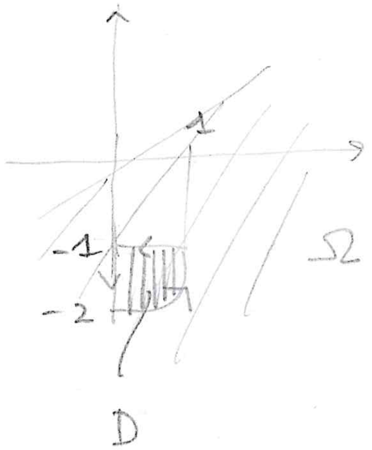
$\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{x-2y}} + \frac{1}{x^2+1} \right) = \frac{1}{(x-2y)^{3/2}}$

$\Big\} \neq$ non è esatta

Procediamo come nell'es 11

$\omega : \{(x,y) \mid y < x/2\} \longrightarrow (\mathbb{R}^2)^*$

$$\omega = \underbrace{\left(\frac{1}{\sqrt{x-2y}} + \frac{1}{x^2+1} \right) dx + \frac{-2}{\sqrt{x-2y}}\, dy}_{\omega_1} + \underbrace{x\,dy}_{\omega_2}$$



ω_1 esatta

$\Downarrow$

$\int_{(\gamma,T)} \omega_1 = 0$

$\downarrow$

Calcoliamo $\int_{(\gamma,T)} \omega_2$ usando G.G.

$$= \iint_D \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(0) \right] dx\,dy = \int_0^1 \left(\int_{x^2-2}^{-1} dy \right) dx =$$

$$= \int_0^1 (-1 - x^2 + 2)\, dx = \cdots$$

13) $\frac{\partial}{\partial y}(\text{sen} z) = 0 = \frac{\partial}{\partial x}(z)$

$\frac{\partial}{\partial z}(\text{sen} z) = \cos z = \frac{\partial}{\partial x} g$

$\frac{\partial}{\partial z} z = 1 = \frac{\partial}{\partial y} g$

$$\Rightarrow \begin{cases} \frac{\partial g}{\partial x} = \cos z \\ \frac{\partial g}{\partial y} = 1 \end{cases}$$

$g(x,y,z) = x \cos z + h(y,z)$

$\downarrow$

$1 = \frac{\partial g}{\partial y} = \frac{\partial h}{\partial y}(y,z) \Rightarrow h(y,z) = y + c(z)$

Allora

$g(x,y,z) = x \cos z + y + c(z) \quad \forall c \in C^\infty(\mathbb{R})$

Primitiva $f : \mathbb{R}^3 \to \mathbb{R}$:

$$\begin{cases} \dfrac{\partial f}{\partial x} = \operatorname{sen} z \\[2mm] \dfrac{\partial f}{\partial y} = z \\[2mm] \dfrac{\partial f}{\partial z} = x \cos z + y + c(z) \end{cases}$$

$$\longrightarrow \quad f(x,y,z) = x \operatorname{sen} z + c_1(y,z)$$

$$z = \frac{\partial f}{\partial y} = \frac{\partial c_1}{\partial y}(y,z) \Rightarrow c_1(y,z) = zy + c_2(z)$$

$$\downarrow$$

$$f(x,y,z) = x \operatorname{sen} z + zy + c_2(z)$$

$$\cancel{x \cos z} + \cancel{y} + c(z) = \frac{\partial f}{\partial z} = \cancel{x \cos z} + \cancel{y} + c_2'(z)$$

$$\Updownarrow$$

$$f(x,y)z) = x \operatorname{sen} z + zy + \int_0^z c(s)\,ds + c_0$$

primitiva di ω ($\forall c \in C^\infty$, $\forall c \in \mathbb{R}$)

Se impongo $f(0,0,0) = 1$

allora $1 = f(0,0,0) \Leftrightarrow c_0 = 1 \Rightarrow f(x,y,z) = x \operatorname{sen} z + zy + \int_0^z c(s)\,ds$
$+ 1$.